Journal of Pure and Applied Algebra 67 (1990) 189–199
North-Holland

ON THE HOMOLOGY OF FREE ABELIANIZED EXTENSIONS OF FINITE GROUPS

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Communicated by K.W. Gruenberg
Received 15 August 1989

Denote by F a free group of finite rank d with normal subgroup R of finite index and let $G = F/R$. By a result of Kuz'min the homology groups $H_n(F/R', \mathbb{Z})$ decompose into the direct sum of a free abelian group of finite rank d_n and a finite abelian group. In this paper the ranks d_n are computed and certain relations are derived. Moreover, it is shown that the Poincaré duality group F/R' is non-orientable if and only if d is even and the 2-Sylow subgroups of G are non-trivial and cyclic.

1. Introduction

Assume we are given a group G by the presentation

$$1 \to R \to F \to G \to 1 \tag{1}$$

with F a free group on a set X. In this note we are interested in the integral homology of the group F/R'. This group is called the *free abelianized extension* of G in view of the exact sequence

$$1 \to R/R' \to F/R' \to G \to 1.$$

We have

$$H_0(F/R', \mathbb{Z}) \simeq \mathbb{Z}, \tag{2}$$

$$H_1(F/R', \mathbb{Z}) \simeq F/F'. \tag{3}$$

In order to examine $H_n(F/R', \mathbb{Z})$ for $n \geq 2$ Kuz'min [6,7] considered the Lyndon–Hochschild–Serre spectral sequence $H_p(F/R, H_q(R/R', \mathbb{Z}))$ which converges to $H_n(F/R', \mathbb{Z})$. Here, as usual, $G = F/R$ acts by conjugation on R/R' and then on $H_q(R/R', \mathbb{Z})$. Since R/R' is free abelian we have the isomorphisms

$$H_q(R/R', \mathbb{Z}) \simeq \Lambda^q(R/R') \quad \text{for } q \geq 0, \tag{4}$$

which are natural with respect to automorphisms of R/R', see [2, Chapter V]. In particular, the isomorphisms (4) are compatible with the G-action. Here $\Lambda^q(R/R')$ denotes the qth exterior power of the free abelian group R/R'.

Kuz'min's main results are collected in the following:

Theorem 1.1. *Assume* $n \geq 2$.

(i) *There is a positive integer* $\tilde{c}_n$ *such that* $\tilde{c}_n \cdot H_i(G, H_n(R/R', \mathbb{Z})) = 0$ *for* $i \geq 1$ *and* $H_0(G, H_n(R/R', \mathbb{Z})) = \tilde{A}_n \oplus \tilde{T}_n$ *with* $\tilde{A}_n$ *free abelian and* $\tilde{c}_n \cdot \tilde{T}_n = 0$.

(ii) *There is a positive integer* c_n *such that*

$$H_n(F/R', \mathbb{Z}) = A_n \oplus T_n \tag{5}$$

with A_n *(non-canonically) isomorphic to* $\tilde{A}_n$ *and* $c_n \cdot T_n = 0$.

(iii) *If* X *and* G *are finite, then the torsion subgoup* T_n *is finite and* A_n *is finitely generated.* □

In this paper we are concerned exclusively with the case mentioned last, i.e. we henceforth assume

(∗) *The finite group* G *is given by the presentation* (1) *with* $F = F(X)$ *a free group of finite rank* $d = |X|$.

We fix the following notation:

$$m = \operatorname{rank} R = 1 + (d-1) \cdot |G|,$$
$$d_n = \operatorname{rank} A_n.$$

Theorem 1.1(ii) may be expanded to $n \in \{0, 1\}$ by (2) and (3). We have:

$$d_0 = 1, \quad T_0 = 0,$$
$$d_1 = d, \quad T_1 = 0.$$

Under the assumption (∗) the group F/R' turns out to be a Poincaré duality group of dimension m. This implies, in particular that $H_n(F/R', \mathbb{Z}) = 0$ for $n > m$. In the following we are interested in getting information about A_n and T_n for $2 \leq n \leq m$.

Theorem 1.2. *Let* $2 \leq n \leq m$. *Then, in the situation* (∗), *we have*

$$d_n = \frac{1}{|G|}\left(\binom{m}{n} + \sum_{1 \neq q \mid n} (-1)^{(q-1)n/q} \cdot b_q \cdot \binom{(m-1)/q}{n/q}\right.$$
$$\left. + \sum_{1 \neq q \mid n-1} (-1)^{(q-1)(n-1)/q} \cdot b_q \cdot \binom{(m-1)/q}{(n-1)/q}\right),$$

where b_q *denotes the number of elements in* G *with order precisely* q. *If the group order* $|G|$ *is prime to* n *and* $n-1$, *then this formula simplifies to* $d_n = (1/|G|) \cdot \binom{m}{n}$.

In case $n = 2$, Theorem 1.2 yields the rank of the free abelian part of $H_2(F/R', \mathbb{Z})$. By Hopf's formula [5] this homology group is isomorphic to $\gamma_2 R/[\gamma_2 R, F] \simeq$

$H_0(G, \gamma_2 R/\gamma_3 R)$; here $\gamma_n R$ denotes the nth term of the lower central series of R, $n \geq 1$. In [12] the rank of the free abelian part of $H_0(G, \gamma_n R/\gamma_{n+1} R)$ was shown to be equal to the rank of $(\gamma_n R/\gamma_{n+1} R)^G$, which was calculated in an earlier paper of Gupta, Laffey and Thomson [4].

Supplementary, Tan [11] gave an estimation of d_2.

As mentioned before, the construction of free abelianized extensions starting from finite G and finite X is a source of Poincaré duality groups. Certain properties are mainly rooted in $G = F/R$. So we may decide the question about orientability of F/R'.

Theorem 1.3. *In the situation* $(*)$ *the Poincaré duality group* F/R' *is non-orientable if and only if* d *is even and the* 2*-Sylow subgroups of* G *are non-trivial and cyclic.*

An essential invariant of a Poincaré duality group Γ is the Euler–Poincaré characteristic:

$$\chi(\Gamma) = \sum_{n=0}^{\operatorname{cd}\Gamma} (-1)^n \cdot \operatorname{rank}_{\mathbb{Z}} H_n(\Gamma, \mathbb{Z}),$$

see [2, Chapter IX]. Here $\operatorname{cd}\Gamma$ denotes the cohomological dimension of Γ. For an abelian group A its rank is defined by $\operatorname{rank}_{\mathbb{Z}} A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$. Since in our situation R/R' is of finite index in F/R' the corresponding characteristics are connected by

$$\begin{aligned}
|F/R' : R/R'| \cdot \chi(F/R') &= |G| \cdot \chi(F/R') = \chi(R/R') \\
&= \sum_{n=0}^{m} (-1)^n \cdot \operatorname{rank}_{\mathbb{Z}} H_n(R/R', \mathbb{Z}) \\
&= \sum_{n=0}^{m} (-1)^n \cdot \operatorname{rank}_{\mathbb{Z}} \Lambda^n(R/R') \\
&= \sum_{n=0}^{m} (-1)^n \cdot \binom{m}{n} = (1-1)^m = 0,
\end{aligned}$$

hence

$$0 = \chi(F/R') = \sum_{n=0}^{m} (-1)^n \cdot d_n. \tag{6}$$

Denote by e the exponent of G. In addition to (6) we state the following relations:

Theorem 1.4. (i) *In the situation* $(*)$ *we have*

$$|G| \cdot \sum_{n=ie+2}^{(i+1)e+1} (-1)^n \cdot d_n = \sum_{n=ie+2}^{(i+1)e+1} (-1)^n \cdot \binom{m}{n} \tag{7}$$

for $0 \leq i < (m-1)/e$.

(ii) *If F/R' is orientable, then*

$$A_n \simeq A_{m-n} \quad (\textit{or, equivalently, } d_n = d_{m-n})$$

and

$$T_n \simeq T_{m-n-1} \quad \textit{for } 0 \leq n \leq m.$$

By summing up all relations of type (7) we obviously get (6).

The arrangement of the paper is as follows. Section 2 has a preliminary character. By simple arguments we prove Kuz'min's Theorem in a weakened form for finite F/R. Then we collect some material concerning symmetric polynomials and derive a character formula for the nth exterior power of a given finite group representation. In Sections 3 and 4 we prove the Theorems 1.2, 1.3 and 1.4 in turn.

2. Preliminaries

2.1. Decomposition of $H_n(F/R', \mathbb{Z})$

For the techniques used in this subsection we refer to [2].

Since R/R' is of finite index in F/R' we have the homological restriction maps

$$\text{res}: H_n(F/R', \mathbb{Z}) \to H_n(R/R', \mathbb{Z})$$

in addition to the corestriction maps

$$\text{cor}: H_n(R/R', \mathbb{Z}) \to H_n(F/R', \mathbb{Z}) \quad \text{for } n \geq 0.$$

Let $a \in \ker \text{res}$. Then $|G| \cdot a = |F/R' : R/R'| \cdot a = \text{cor res } a = \text{cor } 0 = 0$. Hence, ker res is a torsion group of exponent dividing $|G|$.

Write N_G for the norm element of $\mathbb{Z}G: N_G = \sum_{g \in G} g$. As usually, M^G denotes the subgroup of all fixed elements in a G-module M:

$$M^G = \{a \in M \mid {}^g a = a \text{ for all } g \in G\}.$$

We have

$$N_G H_n(R/R', \mathbb{Z}) \subseteq \text{im res} \subseteq H_n(R/R', \mathbb{Z})^G. \tag{8}$$

Indeed, the left inclusion is clear by $N_G H_n(R/R', \mathbb{Z}) = \text{res cor } H_n(R/R', \mathbb{Z})$ and the right inclusion holds since res is a G-module homomorphism and $H_n(F/R', \mathbb{Z})$ is invariant under the G-action.

The zero-dimensional Tate cohomology group $\hat{H}^0(G, H_n(R/R', \mathbb{Z}))$ is of exponent dividing $|G|$. Hence $|G| \cdot H_n(R/R', \mathbb{Z})^G \subseteq \text{im res} \subseteq H_n(R/R', \mathbb{Z})^G$ by (8) and (im res) is free abelian with rank equal to that of $H_n(R/R', \mathbb{Z})^G$. Thus, the exact sequence

$$0 \to \ker \text{res} \to H_n(F/R', \mathbb{Z}) \to \text{im res} \to 0$$

splits and we get

$$H_n(F/R', \mathbb{Z}) = A_n \oplus T_n$$

with

A_n free abelian,
$\operatorname{rank} A_n = \operatorname{rank} H_n(R/R', \mathbb{Z})^G$,
$|G| \cdot T_n = 0$.

Our proof of (5) gives an upper bound for the exponent of T_n which depends on G, whereas Kuz'min's approach yields a universal exponent.

2.2. *The character of the nth exterior power of a finite group representation*

Assume m to be any positive natural number in this subsection. The symmetric group S_m acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_m]$ by canonically permuting the indeterminates. The subring $\mathbb{C}[x_1, \ldots, x_m]^{S_m}$ is the ring of all symmetric polynomials in $x_1, \ldots, x_m$. We refer to [8] for symmetric polynomials.

The following notations will be used ($n \geq 1$):

$x^\varepsilon := x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m}$ and $|\varepsilon| = \varepsilon_1 + \cdots + \varepsilon_m$ for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{N}^m$.
$\alpha^\varepsilon := \alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$, $\varepsilon \in \mathbb{N}^m$.
$\Omega_n := \{\omega \in \mathbb{N}^m \mid \omega_i \in \{0,1\},\ |\omega| = n\}$
$e_n := \sum_{\omega \in \Omega_n} x^\omega$—the nth elementary symmetric polynomial
$\Pi_n := \{\pi \in \mathbb{N}^m \mid |\pi| = n,\ \pi_i = 0 \text{ with exactly one exception}\}$
$p_n := \sum_{\pi \in \Pi_n} x^\pi$—the nth power sum.

Assume $e_0 = 1$.

Now let V be an m-dimensional complex vector space with linearly ordered basis

$$v_1 < v_2 < \cdots < v_m. \tag{9}$$

An endomorphism θ of V induces endomorphisms θ_n of $\Lambda^n V$ for $0 \leq n \leq m$, where $\Lambda^n V$ denotes the nth exterior power of V, by

$$\theta_n(w_1 \wedge \cdots \wedge w_n) := \theta(w_1) \wedge \cdots \wedge \theta(w_n), \tag{10}$$

where $w_i \in V$. Suppose $\Lambda^0 V = \mathbb{C}$, $\theta_0 = \mathrm{id}_{\mathbb{C}}$. If $n \geq 1$, then

$$\mathfrak{B}_n = \{v^\omega := v_1^{\omega_1} \wedge \cdots \wedge v_m^{\omega_m} \mid \omega \in \Omega_n\}$$

is a basis of $\Lambda^n V$. We order the sets Ω_n and $\mathfrak{B}_n$ lexicographically: let $\omega < \omega'$ (and themselves $v^\omega < v^{\omega'}$) if there is an index $i \geq 0$ such that $\omega_1 = \omega_1', \ldots, \omega_i = \omega_i'$ and $\omega_{i+1} < \omega_{i+1}'$. This ordering agrees with (9) in case $n = 1$.

By suitable basis choosing we can reach that θ has Jordan normal form, that means

$$\theta(v_1) = \alpha_1 \cdot v_1, \tag{11}$$

$$\theta(v_i) = \alpha_i \cdot v_i + \gamma_{i-1} \cdot v_{i-1} \quad \text{for } 2 \le i \le m, \tag{12}$$

with $\alpha_i \in \mathbb{C}$, $\gamma_{i-1} \in \{0, 1\}$. Denote by $\operatorname{tr} \theta$ the trace of θ.

Lemma 2.1. *If $v^\omega \in \mathfrak{B}_n$, then*

(i) $$\theta_n(v^\omega) = \alpha^\omega \cdot v^\omega + \sum_{\delta<\omega} \gamma(\delta) \cdot v^\delta \tag{13}$$

with suitable $\gamma(\delta) \in \mathbb{C}$.

(ii) $\operatorname{tr} \theta_n = e_n(\alpha_1, \dots, \alpha_m)$.

Proof. (i) We proceed by induction on n. If $n=1$, then (13) follows from (11) and (12).

Now suppose that for all $\varrho \in \Omega_{n-1}$ we have $\theta_{n-1}(v^\varrho) = \alpha^\varrho \cdot v^\varrho + \sum_{\tau<\varrho} \gamma(\tau) \cdot v^\tau$. Consider (13). Define $j := \max\{i \mid \omega_i = 1\}$. If $j=1$, then $n=1$ and our assertion holds. So assume $j \ge 2$. Then

$$\omega = (\omega_1, \dots, \omega_{j-1}, 1, 0, \dots, 0) \quad \text{and}$$

$$\varrho = (\omega_1, \dots, \omega_{j-1}, 0, 0, \dots, 0) \in \Omega_{n-1}.$$

Hence

$$\begin{aligned}
\theta_n(v^\omega) &= (\theta_{n-1}(v^\varrho)) \wedge (\theta(v_j)) \\
&= \left(\alpha^\varrho \cdot v^\varrho + \sum_{\tau<\varrho} \gamma(\tau) \cdot v^\tau\right) \wedge (\alpha_j \cdot v_j + \gamma_{j-1} \cdot v_{j-1}) \\
&= \alpha^\omega \cdot v^\omega + \alpha^\varrho \cdot \gamma_{j-1} \cdot (v^\varrho \wedge v_{j-1}) \\
&\quad + \sum_{\tau<\varrho} \gamma(\tau) \cdot \alpha_j \cdot (v^\tau \wedge v_j) + \sum_{\tau<\varrho} \gamma(\tau) \cdot \gamma_{j-1} \cdot (v^\tau \wedge v_{j-1}).
\end{aligned}$$

Now $v^\varrho \wedge v_{j-1} \ne 0$ if and only if $\omega_{j-1} = 0$. But in this case $v^\varrho \wedge v_{j-1} < v^\omega$.

Similar arguments are applicable to the remaining summands. This provides (i).

(ii) is an immediate consequence of (i). □

Now we pass on to a finite group G and a representation φ of G on V with character χ. Then φ induces a representation φ_n on $\Lambda^n V$ for $n \ge 0$ accordingly to (10). The corresponding character is denoted χ_n. The preceding lemma may be adapted to this situation. We get the following assertion:

Corollary 2.2. $\chi_n(g) = e_n(\alpha_{1,g}, \dots, \alpha_{m,g})$ *with* $\alpha_{1,g}, \dots, \alpha_{m,g} \in \mathbb{C}$ *the eigenvalues of* $\varphi(g)$. □

3. The rank formula

Our aim in this section is to compute $d_n = \operatorname{rank} A_n$ in situation (*). Recall that a *G-lattice* is a G-module with underlying free abelian group of finite rank. Clearly, all exterior powers of R/R' are G-lattices.

Denote by φ that representation of G on $V = R/R' \otimes \mathbb{C}$ which is induced by G-conjugation on R/R'.

The rank d_n of A_n may be computed as the dimension of the subspace $(\Lambda^n V)^G$. Indeed, $\Lambda^n(R/R')$ is a G-lattice and so we have

$$(\Lambda^n(R/R'))^G = \ker\left(\Lambda^n(R/R') \xrightarrow{N_G - |G|} \Lambda^n(R/R')\right).$$

Hence,

$$\begin{aligned}\Lambda^n(R/R')^G \otimes \mathbb{C} &= \ker\left(\Lambda^n(R/R') \otimes \mathbb{C} \xrightarrow{N_G - |G|} \Lambda^n(R/R') \otimes \mathbb{C}\right)\\ &= \Lambda^n(R/R' \otimes \mathbb{C})^G = (\Lambda^n V)^G\end{aligned}$$

and consequently, $d_n = \dim (\Lambda^n V)^G$.

Since $\Lambda^n V$ is a direct sum of irreducible complex representations of G, it follows that $\dim (\Lambda^n V)^G$ is the number of times the trivial one-dimensional representation 1_G arises in this direct decomposition, hence

$$d_n = \dim (\Lambda^n V)^G = \langle \chi_n, 1_G \rangle = \frac{1}{|G|} \cdot \sum_{g \in G} \chi_n(g), \tag{14}$$

where $\langle\, , \rangle$ is the scalar product at the vector space of complex functions defined on G, see [9, Chapter I].

We know the character $\chi = \chi_1$ of $V = \Lambda^1 V$ by a result of Gaschütz [3]:

$$\chi(g) = \begin{cases} m & \text{if } g = 1, \\ 1 & \text{if } g \neq 1. \end{cases}$$

We first calculate the induced character χ_n on $\Lambda^n V$.

Lemma 3.1. (i) *Let $2 \leq n \leq m$, $g \in G \setminus \{1\}$ and $q = \operatorname{ord} g$. Then*

$$\chi_n(g) = (-1)^{n(q-1)/q} \cdot \binom{(m-1)/q}{n/q} \qquad \text{if } n \equiv 0 \bmod q,$$

$$\chi_n(g) = (-1)^{(n-1)(q-1)/q} \cdot \binom{(m-1)/q}{(n-1)/q} \qquad \text{if } n \equiv 1 \bmod q,$$

and $\chi_n(g) = 0$ otherwise.

(ii) $\chi_m(g) = (-1)^{(q-1)(d-1)|G|/q}$;

(iii) *If $n \leq m$, then $\chi_n(1) = \binom{m}{n}$.*

Proof. We use the means of generating functions associated with the elementary symmetric polynomials and the power sums:

$$E_g(y) = \sum_{n\geq 0} e_n(\alpha_{1,g}, \dots, \alpha_{m,g}) \cdot y^n,$$

$$P_g(y) = \sum_{n\geq 1} p_n(\alpha_{1,g}, \dots, \alpha_{m,g}) \cdot y^{n-1},$$

where $\alpha_{1,g}, \dots, \alpha_{m,g}$ are the eigenvalues of g acting on V. By Corollary 2.2 we get

$$E_g(y) = \sum_{n\geq 0} \chi_n(g) \cdot y^n. \tag{15}$$

Gaschütz's result together with the identities

$$\chi(g^n) = \operatorname{tr} \varphi(g^n) = \operatorname{tr} (\varphi(g))^n = p_n(\alpha_{1,g}, \dots, \alpha_{m,g})$$

yield

$$\begin{aligned} P_g(y) &= \sum_{\substack{n\geq 1\\ q\mid n}} m \cdot y^{n-1} + \sum_{\substack{n\geq 1\\ q\nmid n}} y^{n-1} \\ &= \frac{(m-1)\cdot y^{q-1}}{1-y^q} + \frac{1}{1-y}. \end{aligned}$$

Hence

$$P_g(-y) = \frac{\mathrm{d}}{\mathrm{d}y} \log[(1-(-y)^q)^{(m-1)/q} \cdot (1+y)].$$

The identity

$$P_g(-y) = \frac{\mathrm{d}}{\mathrm{d}y} \log(E_g(y)),$$

for which we refer to [8, 2.10′], provides

$$E_g(y) = c_g \cdot (1-(-y)^q)^{(m-1)/q} \cdot (1+y) \tag{16}$$

with a suitable $c_g \in \mathbb{C}$. By setting $y=0$ we get $c_g = 1$. The assertions of the lemma follow by comparing the coefficients in (15) and (16). □

Proof of Theorem 1.2. With the help of Lemma 3.1 we get from (14)

$$\begin{aligned} d_n = \frac{1}{|G|} \cdot \Bigg(\binom{m}{n} &+ \sum_{1\neq q\mid n} \sum_{\substack{g\in G\\ \operatorname{ord} g = q}} (-1)^{(q-1)n/q} \cdot \binom{(m-1)/q}{n/q} \\ &+ \sum_{1\neq q\mid n-1} \sum_{\substack{g\in G\\ \operatorname{ord} g = q}} (-1)^{(q-1)(n-1)/q} \cdot \binom{(m-1)/q}{(n-1)/q} \Bigg) \end{aligned}$$

for $2 \leq n \leq m$. If b_q denotes the number of elements in G with order precisely q, then

$$\begin{aligned} d_n = \frac{1}{|G|} \Bigg(\binom{m}{n} &+ \sum_{1\neq q\mid n} (-1)^{(q-1)n/q} \cdot b_q \cdot \binom{(m-1)/q}{n/q} \\ &+ \sum_{1\neq q\mid n-1} (-1)^{(q-1)(n-1)/q} \cdot b_q \cdot \binom{(m-1)/q}{(n-1)/q} \Bigg). \end{aligned}$$

□

4. Poincaré duality

Recall [2, Chapter VIII] that a group Γ of finite cohomological dimension k and of type FP is termed a *Poincaré duality group of dimension* k (PD^k) if $D := H^k(\Gamma, \mathbb{Z}\Gamma)$ is isomorphic to $\mathbb{Z}$ (as an abelian group) and $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq k$.

The abelian group D is a left Γ-module by reversing the right regular Γ-action on $\mathbb{Z}\Gamma$. Then for an arbitrary left Γ-module M, cap multiplication with the fundamental class $z \in H_k(\Gamma, D)$ yields the isomorphisms

$$H^n(\Gamma, M) \simeq H_{k-n}(\Gamma, D \otimes M),$$

where $D \otimes M$ is endowed with the left diagonal Γ-action.

If the Γ-action on D turns out to be trivial, then Γ is called *orientable*, otherwise *non-orientable*.

Recall that $H_k(\Gamma, \mathbb{Z})$ denotes the homology of Γ with coefficients in the *trivial* Γ-module $\mathbb{Z}$ (in contrast with $H_k(\Gamma, D)$).

Proof of Theorem 1.3. In our situation (*) the group F/R' as a torsion-free (see [10]) extension of the Poincaré duality group R/R' of dimension m is itself a PD^m, see [2, Chapter VIII] or [1, Chapter III].

Since $H_m(F/R', \mathbb{Z}) = \mathbb{Z}$ if F/R' is orientable and $H_m(F/R', \mathbb{Z}) = 0$ otherwise (see [1, p. 174]), we claim that $d_m = 0$ is equivalent to the non-orientability of F/R'. Moreover, by (14) and Lemma 3.1(ii)

$$d_m = \frac{1}{|G|} \cdot \sum_{g \in G} \chi_m(g) \in \{0, 1\}$$

and d_m vanishes if and only if there is an element $g \in G$ with $\chi_m(g) = -1$. Again by Lemma 3.1(ii) this condition is equivalent to $2 \mid d$ and the existence of an element g of even order and odd index in G. This gives Theorem 1.3. □

Proof of Theorem 1.4. For $0 \leq i < (m-1)/e$ define

$$K_i(g) := \sum_{n=ie+2}^{(i+1)e+1} (-1)^n \cdot \chi_n(g). \tag{17}$$

Then

$$K_i(1) := \sum_{n=ie+2}^{(i+1)e+1} (-1)^n \cdot \binom{m}{n} \tag{18}$$

by Lemma 3.1(iii).

Now let $g \neq 1$, and q denote the order of g. Let $rq = e$. Then

$$K_i(g) = \sum_{j=1}^{r} \sum_{n=ie+(j-1)q+2}^{ie+jq+1} (-1)^n \cdot \chi_n(g)$$

and the summation index of the inner sum covers a full system of residue classes modulo q. By Lemma 3.1 the summands corresponding to $n=ie+jq$ and $n=ie+jq+1$ are the only ones which do not vanish. Hence

$$\begin{aligned}
K_i(g) &= \sum_{j=1}^{r} (-1)^{ie+jq} \cdot \chi_{ie+jq}(g) \\
&\quad + \sum_{j=1}^{r} (-1)^{ie+jq+1} \cdot \chi_{ie+jq+1}(g) \\
&= \sum_{j=1}^{r} (-1)^{ie+jq} \cdot \left((-1)^{(ir+j)(q-1)} \cdot \binom{(m-1)/q}{ir+j} \right. \\
&\qquad \left. + (-1)^{1+(ir+j)(q-1)} \cdot \binom{(m-1)/q}{ir+j} \right)
\end{aligned}$$

and

$$K_i(g) = 0 \quad \text{if } g \neq 1 \tag{19}$$

follows. We obtain

$$\begin{aligned}
|G| \cdot \sum_{n=ie+2}^{(i+1)e+1} (-1)^n \cdot d_n &= \sum_{n=ie+2}^{(i+1)e+1} \sum_{g \in G} (-1)^n \cdot \chi_n(g) && \text{by (14)} \\
&= \sum_{g \in G} K_i(g) && \text{by (17)} \\
&= \sum_{n=ie+2}^{(i+1)e+1} (-1)^n \cdot \binom{m}{n} && \text{by (18) and (19).}
\end{aligned}$$

This proves the first part of Theorem 1.4.

Assertion (ii) of Theorem 1.4 is true for any oriented Poincaré duality group Γ of dimension m. This is a consequence of the Universal Coefficients Theorem for group (co)homology (see [2]). The equality $d_n = d_{m-n}$ may also be shown by simple manipulations with the rank formula appearing in Theorem 1.2. □

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